1993

On Model Identification of Gaussian Reciprocal Processes from the Eigenstructure of their Covariances

Borges, Carlos F.

On Model Identification of Gaussian Reciprocal Processes from the Eigenstructure of their Covariances

Appears in:
Computation and Control III: Proceedings of the Third Bozeman Conference, (J. Lund and K. Bowers, eds.), Progress in Systems and Control Theory, Birkhauser, (1993), pp. 63–72.

Carlos F. Borges
Code Ma/Bc
Naval Postgraduate School
Monterey, CA 93943

Ruggero Frezza
DEI, Univ. di Padova
via Gradenigo 6/A
35131 PADOVA - ITALY

We present a numerical algorithm to reconstruct models of scalar Gaussian reciprocal processes from the eigenstructure of their covariances. This also fills a gap in the inverse eigenproblems for Jacobi matrices such as those given by [3, 8, 9] and others. Because of its properties the algorithm can be extended to other classes of matrices. We show its application to the important class of symmetric arrow matrices.

1 Models of Gaussian reciprocal processes

A stochastic process $x(k)$ defined on $[1, N]$ is *reciprocal* if for any subinterval $[l, m]$ of $[1, N]$ the process in the interior of $[l, m]$ is conditionally independent of the process in $[1, N] - [l, m]$ given $x(l)$ and $x(m)$. For a more rigorous definition see [10]. Reciprocal processes generalize Markov processes since a Markov process is reciprocal while the converse, in general, is not true, see [10] for an example of a process which is reciprocal and not Markov.

Recently, it was shown [11, 12, 5] that, under suitable assumptions, a discrete time Gaussian reciprocal process $x(k)$ satisfies a nearest neighbor model like the following

$$m_0(k)x(k) - m_-(k)x(k-1) - m_+(k)x(k+1) = \xi(k) \qquad (1.1)$$

where $\xi(k)$ is a zero mean Gaussian process with covariance

$$\begin{aligned} E[\xi(k)\xi(k)] &= m_0(k) \\ E[\xi(k)\xi(k+1)] &= -m_-(k+1) = -m_+(k) \\ E[\xi(k)\xi(k+l)] &= 0 \quad \text{for } l > 1 \end{aligned} \qquad (1.2)$$

In matrix form the model (1.1) can be written as

$$\mathrm{A}\mathbf{x} = \xi \qquad (1.3)$$

where

$$\mathbf{x}^T = \begin{bmatrix} x(1) & x(2) & \cdots & x(N) \end{bmatrix}, \tag{1.4}$$

$$\xi^T = \begin{bmatrix} \xi(1) & \xi(2) & \cdots & \xi(N) \end{bmatrix}, \tag{1.5}$$

and A is the following Jacobi matrix

$$A = \begin{bmatrix} m_0(1) & -m_+(1) & & & \\ -m_+(1) & m_0(2) & -m_+(2) & & \\ & -m_+(2) & & \ddots & \\ & & \ddots & & -m_+(n-1) \\ & & & -m_+(n-1) & m_0(n) \end{bmatrix}$$

The covariance structure (1.2) of the noise process corresponds to

$$E[\xi\xi^T] = A. \tag{1.6}$$

From (1.3) and (1.6) we see that

$$E[\mathbf{x}\xi^T] = \mathbf{I} \tag{1.7}$$

and, therefore, that

$$A\mathbf{R} = \mathbf{I} \tag{1.8}$$

where $\mathbf{R} = E[\mathbf{x}\mathbf{x}^T]$. Thus, the matrix A characterizing the model of the reciprocal process x and the covariance $\mathbf{R}$ of $\mathbf{x}$ have a related eigenstructure. If $(\lambda_k, \mathbf{u}_k)$ are the eigenpairs of A then $(1/\lambda_k, \mathbf{u}_k)$ are the eigenpairs of $\mathbf{R}$.

This leads us to consider the possibility of identifying the reciprocal model (1.1) of such a process starting from the eigenfunctions of its covariance. Clearly, this is equivalent to reconstructing the matrix A from its eigenstructure and this is a well known problem in the literature see, for example, [8], [9] and references therein.

We propose an algorithm not yet studied in the literature on reconstructing Jacobi matrices. We reconstruct A from its two extremal eigenpairs $(\lambda_1, \mathbf{u}_1)$ and $(\lambda_n, \mathbf{u}_n)$. We show that the algorithm is well posed. The extremal eigenpairs also have the advantage that they can be easily computed from the covariance $\mathbf{R}$ using Krylov sequence methods like the Lanczos algorithm or power and inverse iteration [7].

The algorithm is straightforward and can be generalized to other classes of matrices. For example, we show how to reconstruct symmetric arrow matrices. We conjecture that this inverse problem is well-posed for all unreduced symmetric acyclic matrices which represent an interesting generalization of the reciprocal models.

The algorithm also serves to identify Markov models. In [12], it was shown that a Markov process $x(k)$ satisfying the model

$$\Omega \mathbf{x} = B\mathbf{w} \tag{1.9}$$

where

$$\Omega = \begin{bmatrix} 1 & & & & \\ -a(1) & 1 & & & \\ & -a(2) & \ddots & & \\ & & \ddots & & \\ & & & -a(n-1) & 1 \end{bmatrix} \tag{1.10}$$

$$B = \mathrm{diag}(b(k)) \tag{1.11}$$

and

$$\mathbf{w} = \begin{bmatrix} w(0) & w(1) & \cdots & w(N-1) \end{bmatrix} \tag{1.12}$$

where $w(k)$ are Gaussian, zero mean, independent random variables with unitary variance, also satisfies a reciprocal model like (1.3) where

$$\Lambda = \Omega^T Q^{-1} \Omega \tag{1.13}$$

with $Q = BB^T$ and

$$\xi = \Omega^T Q^{-1} \mathbf{w}. \tag{1.14}$$

Therefore, Ω and B can be obtained from Λ by performing a Cholesky factorization.

In practice, the covariance $\mathbf{R}$ will be corrupted by noise; we will know the covariance $\mathbf{R}_y$ of the observations

$$\mathbf{y} = C\mathbf{x} + \mathbf{v} \tag{1.15}$$

where $C = \mathrm{diag}_{N \times N}(c)$. If $v(k)$ are independent Gaussian random variables identically distributed with zero mean and variance v, then $\mathbf{R}_y$ is related to $\mathbf{R}$ by

$$\mathbf{R}_y = C\mathbf{R}C^T + vI. \tag{1.16}$$

One can show that as long as $v \neq 0$ the covariance $\mathbf{R}_y$ does not have a tridiagonal inverse. We can refine the algorithm so that given $\mathbf{R}_y$ it estimates v and Λ such that

$$\Lambda(\mathbf{R}_y - vI) = \mathrm{I}. \tag{1.17}$$

2 Reconstructing a Jacobi Matrix

Let T be an unreduced $n \times n$ real symmetric tridiagonal matrix

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & & \ddots & \\ & & \ddots & & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix} \tag{2.18}$$

with $\beta_i > 0$ for $i = 1, 2, ..., n-1$. Using notation introduced in [15] we say $T \in \mathbf{UST}_+(n)$.

We want to reconstruct T from two of its eigenpairs $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$. The eigenvector recurrence for symmetric tridiagonal matrices is

$$\beta_{i-1}u_{i-1} + \alpha_i u_i + \beta_i u_{i+1} = \lambda u_i \tag{2.19}$$

where u_i is the i'th element of $\mathbf{u}$, and we define $\beta_0 \equiv \beta_n \equiv 0$. Applying this relation to both eigenpairs gives

$$\beta_{i-1}u_{i-1} + \alpha_i u_i + \beta_i u_{i+1} = \lambda u_i \tag{2.20}$$
$$\beta_{i-1}v_{i-1} + \alpha_i v_i + \beta_i v_{i+1} = \mu v_i \tag{2.21}$$

and, clearly

$$\begin{bmatrix} u_i & u_{i+1} \\ v_i & v_{i+1} \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} = \begin{bmatrix} \lambda u_i - \beta_{i-1}u_{i-1} \\ \mu v_i - \beta_{i-1}v_{i-1} \end{bmatrix}. \tag{2.22}$$

This is a *forward* recurrence for the elements of T (there is an analogous backward recurrence). Setting $i = 1$ in 2.22 and solving gives the initial condition

$$\alpha_1 = \frac{\lambda u_1 v_2 - \mu v_1 u_2}{u_1 v_2 - v_1 u_2}, \tag{2.23}$$
$$\beta_1 = \frac{\mu - \lambda}{u_1 v_2 - v_1 u_2} u_1 v_1. \tag{2.24}$$

The terminal condition can be found in a similar manner

$$\alpha_n = \frac{\mu v_n u_{n-1} - \lambda u_n v_{n-1}}{u_{n-1}v_n - v_{n-1}u_n}, \quad (2.25)$$
$$\beta_{n-1} = \frac{\lambda - \mu}{u_{n-1}v_n - v_{n-1}u_n} u_n v_n. \quad (2.26)$$

Applying Cramer's rule to 2.22 we get

$$\alpha_i = \frac{\lambda u_i v_{i+1} - \mu v_i u_{i+1} + \beta_{i-1}(v_{i-1}u_{i+1} - u_{i-1}v_{i+1})}{u_i v_{i+1} - v_i u_{i+1}} \quad (2.27)$$
$$\beta_i = \frac{(\mu - \lambda)u_i v_i + \beta_{i-1}(u_{i-1}v_i - v_{i-1}u_i)}{u_i v_{i+1} - v_i u_{i+1}} \quad (2.28)$$

Combining equations 2.28 and 2.24 it follows that

$$\beta_i = \frac{\mu - \lambda}{u_i v_{i+1} - v_i u_{i+1}} \sum_{k=1}^{i} u_k v_k \quad (2.29)$$

for $i = 1, 2, ..., n-1$. The backward formula

$$\beta_i = \frac{\lambda - \mu}{u_i v_{i+1} - v_i u_{i+1}} \sum_{k=i+1}^{n} u_k v_k. \quad (2.30)$$

follows directly from 2.29 and the orthogonality of $\mathbf{u}$ and $\mathbf{v}$.

The cost of reconstructing the β_i is $7n-8$ flops, and an additional $9n-10$ flops for the α_i given that the computation is properly implemented.

The reconstruction formulas require that the denominators of 2.28 and 2.27 are not zero, this is always the case if the two eigenpairs in question are the extremal ones. We introduce the following theorem from [14]

Theorem 2.1 *If $T \in \mathbf{UST}_+(n)$ then the number of sign changes between consecutive elements of the k'th eigenvector of T, denoted s_k, is k.*

We refer the reader to [14] for a proof but note that it follows from the Sturm sequence property for the characteristic polynomials of the principal minors. With this fact we prove the following theorem.

Theorem 2.2 *If $T \in \mathbf{UST}_+(n)$ and if $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$ are the largest and smallest eigenpairs of T, respectively, then $u_i v_{i+1} - v_i u_{i+1} \neq 0$ for $i = 1, 2, ..., n-1$.*

Proof. Since v_i and v_{i+1} have opposite sign and u_i and u_{i+1} have the same sign it follows that both terms in $u_i v_{i+1} - v_i u_{i+1}$ have the same sign. Moreover, the strict interlacing property for unreduced symmetric tridiagonals (see [16] p. 300) guarantees that all terms are nonzero, and the theorem follows. ■

Hence, using the two extremal eigenpairs of $T \in \mathbf{UST}_+$ we can always reconstruct the original matrix using formulas 2.27, 2.29, and 2.30. The restriction that $\beta_i > 0$ is an artificial one that simplifies the proofs and may be relaxed to simple unreducedness, $\beta_i \neq 1$. Finally, note that the denominator is computed without cancellation because of the sign pattern.

3 Reconstructing the Arrow Matrix

To demonstrate the generality of this approach we now show how to reconstruct the arrow matrix in a similar manner. The arrow is of some importance as it occurs in certain parallel divide and conquer schemes for solving eigenproblems [1] and also occurs in certain problems of physics [13].

The general form of an arrow matrix is

$$T = \begin{bmatrix} \alpha_1 & & & & \beta_1 \\ & \alpha_2 & & & \beta_2 \\ & & \ddots & & \vdots \\ & & & \alpha_{n-1} & \beta_{n-1} \\ \beta_1 & \beta_2 & & \beta_{n-1} & \gamma \end{bmatrix} \tag{3.31}$$

If $\beta_i > 0$ for $i = 1, 2, ..., n-1$ then we shall say that $T \in \mathbf{USA}_+(n)$, where $\mathbf{USA}_+(n)$ is the set of unreduced symmetric arrow matrices with $\beta_i > 0$. Proceeding as before, we let $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$ be two eigenpairs of T. The eigenvector recurrence yields

$$\begin{bmatrix} u_i & u_n \\ v_i & v_n \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} = \begin{bmatrix} \lambda u_i \\ \mu v_i \end{bmatrix} \tag{3.32}$$

for $i = 1, 2, ..., n-1$. Applying Cramer's rule gives

$$\alpha_i = \frac{\lambda v_n u_i - \mu u_n v_i}{v_n u_i - u_n v_i} \tag{3.33}$$

$$\beta_i = \frac{u_i v_i (\mu - \lambda)}{v_n u_i - u_n v_i} \tag{3.34}$$

Moreover, the eigenvector relation also gives

$$\gamma = \lambda - \frac{1}{u_n} \sum_{i=1}^{n-1} \beta_i u_i \tag{3.35}$$

This gives a simple, parallel reconstruction algorithm. The only question is whether or not the determinants $v_n u_i - u_n v_i$ are all nonzero. To demonstrate that they are, under the correct conditions, we need to establish some facts about the eigenvectors of an unreduced arrow matrix. We begin by noting that

$$T - \lambda I = \begin{bmatrix} A - \lambda I & \mathbf{b} \\ \mathbf{b}^T & \gamma - \lambda \end{bmatrix} \tag{3.36}$$

where $A = \operatorname{diag}(\alpha_1, \alpha_2, ..., \alpha_{n-1})$, and $\mathbf{b} = [\beta_1, \beta_2, ..., \beta_{n-1}]^T$. Provided that the α_i are all distinct the Gauss factorization of T is

$$\begin{bmatrix} A - \lambda I & \mathbf{b} \\ \mathbf{b}^T & \gamma - \lambda \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathbf{b}^T (A - \lambda I)^{-1} & 1 \end{bmatrix} \begin{bmatrix} A - \lambda I & \mathbf{b} \\ 0^T & -f(\lambda) \end{bmatrix} \tag{3.37}$$

where f, the *spectral function*, is a rational Pick function and is given by

$$f(\lambda) = \lambda - \gamma + \sum_{i=1}^{n-1} \frac{\beta_i^2}{\alpha_i - \lambda} \tag{3.38}$$

It is obvious from equations 3.37 and 3.38 that the zeros of f are the eigenvalues of T and that the α_i interlace the eigenvalues. Henceforth, we shall consider only those elements of $\mathbf{USA}_+(n)$ with distinct α_i. The eigenvector associated with a given eigenvalue λ is

$$\mathbf{u}(\lambda) = \begin{bmatrix} (\lambda I - A)^{-1} \mathbf{b} \\ 1 \end{bmatrix} \tag{3.39}$$

Combining this description of the eigenvectors with the fact that the α_i interlace the eigenvalues, it is easy to verify the following theorem

Theorem 3.1 *Let $T \in \mathbf{USA}_+(n)$ have distinct elements α_i, and order its eigenvalues, λ_i, so that $\lambda_1 > \lambda_2 > ... > \lambda_n$. Then, the eigenvectors, $\mathbf{u}(\lambda_k)$, from 3.39 satisfy*
1. $\mathbf{u}_i(\lambda_k) \neq 0$ for any $i = 1, 2, ..., n$.
2. $\mathbf{u}(\lambda_k)$ has precisely $k-1$ elements that are less than zero, and $n-k+1$ elements that are greater than zero.

Proof. The proof follows directly from formula 3.39, the interlacing property, and the positivity of the β_i. ■

This simplifies the reconstruction formulas since, if we normalize the eigenvectors so that their last elements are one, they become

$$\alpha_i = \frac{\lambda u_i - \mu v_i}{u_i - v_i} \tag{3.40}$$

$$\beta_i = \frac{(\mu - \lambda) u_i v_i}{u_i - v_i} \tag{3.41}$$

$$\gamma = \mu - \sum_{i=1}^{n-1} \beta_i v_i \tag{3.42}$$

With these formulas, we can reconstruct the arrow matrix in $8n-7$ flops. To establish well-posedness we must verify that none of the denominators in these formulas are zero. If we use the extremal eigenpairs, then this follows directly from fact 2 of theorem 3.1. However, for the symmetric arrow matrix a more general result is possible.

Theorem 3.2 *Let $T \in \mathbf{USA}_+(n)$ have distinct α_i. If λ and μ are distinct eigenvalues of T then $\mathbf{u}_i(\lambda) \neq \mathbf{u}_i(\mu)$.*

Proof. From 3.39 we have

$$\mathbf{u}_i(\lambda) = \frac{\beta_i}{\lambda - \alpha_i} \neq \frac{\beta_i}{\mu - \alpha_i} = \mathbf{u}_i(\mu) \tag{3.43}$$

∎

The reconstruction algorithm has another very important property if the two extremal eigenpairs are used, then the β_i can be found, up to the scaling factor $\lambda_n - \lambda_1$, without cancellation. This follows from the fact that if the corresponding eigenvectors are eigenvectors corresponding to the two extremal eigenvalues of T, and if they are normalized so that their last elements are both ones, then all of their remaining elements must have opposite signs. This is fortuitous since it means that the differences that appear in the denominator do not involve cancellation. Moreover, if T is indefinite there are no cancellations whatsoever in computing the β_i. Conversely, if T is definite there are no cancellations in computing the α_i. If T is semi-definite then there is no cancellation at all (including the computation of γ). The computation of γ involves one cancellation if the matrix is indefinite, and none if it is definite, or semi-definite, provided we choose the correct eigenvector for its computation. In any case, whenever there is cancellation in this algorithm, it is benign.

References

[1] P. Arbenz, *Divide and conquer algorithms for the bandsymmetric eigenvalue problem*, in Parallel Computing '91, D. J. Evans, G. R. Joubert, and H. Liddell, eds., Elsevier Science Publishers B. V., Amsterdam, 1992, pp. 151–158.

[2] R. Ash and M. Gardner, *Topics in Stochastic Processes*, Series on Probability and Mathematical Statistics, Academic Press, 1975.

[3] D. Boley and G. Golub, *A modified method for reconstructing periodic Jacobi matrices*, Math. Comp., 42 (1984), pp. 143–150.

[4] C. Borges, R. Frezza, and W. Gragg, *Some inverse eigenproblems for Jacobi and arrow matrices*, J. Numer. Linear Algebra Appl., (1992). In review.

[5] R. Frezza, *Modeling of Higher Order and Mixed Order Gaussian Reciprocal Processes with Application to the Smoothing Problem*, PhD thesis, University of California, Davis, 1990.

[6] I. Gel'fand and B. Levitan, *On the determination of a differential equation from its spectral function*, AMS Translations, 2 (1955), pp. 253–304.

[7] G. Golub and C. V. Loan, *Matrix Computations*, The Johns Hopkins University Press, 1983.

[8] W. Gragg and W. Harrod, *The numerically stable reconstruction of jacobi matrices from spectral data*, Numer. Math., 44 (1984), pp. 317–335.

[9] W. Hegland and J. Marti, *Algorithms for the reconstruction of special Jacobi matrices from their eigenvalues*, SIAM J. Matrix Anal. Appl., 10 (1989), pp. 219–228.

[10] B. Jamison, *Reciprocal processes: The stationary Gaussian case*, Ann. of Math. Stat., 41 (1970), pp. 1624–1630.

[11] A. Krener, R. Frezza, and B. Levy, *Gaussian reciprocal processes and self adjoint stochastic differential equations of second order*, Stochastics and Stochastics Reports, 34 (1991), pp. 29–56.

[12] B. Levy, R. Frezza, and A. Krener, *Modeling and estimation of discrete time Gaussian reciprocal processes*, IEEE Trans. Automat. Contr., AC–35 (1990), pp. 1013–1023.

[13] D. P. O'Leary and G. W. Stewart, *Computing the eigenvalues and eigenvectors of arrowhead matrices*, J. Comp. Phys., 90 (1990), pp. 497–505.

[14] B. Parlett, *The Symmetric Eigenvalue Problem*, Prentice–Hall, Englewood Cliffs, NJ, 1980.

[15] B. Parlett and W.-D. Wu, *Eigenvector matrices of symmetric tridiagonals*, Numer. Math., 44 (1984), pp. 103–110.

[16] J. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.